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A solvable master equation for population inversion

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Abstract. The master equation derived for population inversion in a three-level model is solved exactly. The time-dependent behaviour of the process can then be discussed.

Recently, in discussing the problem of the double-reservoir and negative temperature, a master equation has been derived for the three-level model consisting of N three-level atoms embedded in a crystal lattice heat bath and interacting with a pump lamp heat bath (Zheng and Peng 1982). In terms of the boson representation operators $a_j^{(\nu)+}, a_j^{(\nu)}$ for a three-level atom (Rai and Mehta 1982), the creation and annihilation operators for pumping field photons of mode α and those for lattice phonons of mode β, the Hamiltonian can be written as

$$H = H_0 + V, \tag{1a}$$

$$H_0 = H_A + H_c + H_p = \sum_{\nu=1}^{N} \sum_{j=1}^{3} a_j^{(\nu)+} a_j^{(\nu)} \hbar\omega_j + \sum_\alpha \hbar\Omega_p^{(\alpha)} \mathcal{O}_p^{(\alpha)+} \mathcal{O}_p^{(\alpha)} + \sum_\beta \hbar\Omega_c^{(\beta)} \mathcal{O}_c^{(\beta)+} \mathcal{O}_c^{(\beta)} \tag{1b}$$

with

$$\omega_3 > \omega_2 > \omega_1,$$

$$V = \sum_{\nu=1}^{N} \sum_\alpha g_\alpha (a_3^{(\nu)+} a_1^{(\nu)} + a_1^{(\nu)+} a_3^{(\nu)})(\mathcal{O}_p^{(\alpha)+} + \mathcal{O}_p^{(\alpha)})$$
$$+ \sum_{\nu=1}^{N} \sum_\beta g_\beta (a_2^{(\nu)+} a_2^{(\nu)} + a_1^{(\nu)+} a_2^{(\nu)})(\mathcal{O}_c^{(\beta)+} + \mathcal{O}_c^{(\beta)}) \tag{1c}$$

where we concentrate our attention only on the formation of population inversion.

Under the standard approximation that the relaxation times of the phonon bath and photon bath are much smaller than that of the atomic system, we can always take

$$\langle \mathcal{O}_p^{(\alpha)+} \mathcal{O}_p^{(\alpha)} \rangle = \bar{n}_p^{(\alpha)} = [\exp(\beta_p \hbar\Omega_p^{(\alpha)}) - 1]^{-1}, \tag{2a}$$

$$\langle \mathcal{O}_c^{(\beta)+} \mathcal{O}_c^{(\beta)} \rangle = \bar{n}_c^{(\beta)} = [\exp(\beta_c \hbar\Omega_c^{(\beta)}) - 1]^{-1}, \tag{2b}$$

where β_p, β_c are the reciprocal temperatures of the reservoirs.

By calculating the transition probabilities with

$$w = (2\pi/\hbar) |\langle f | V | i \rangle|^2 \delta(E_f^0 - E_i^0) \tag{3}$$

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Pauli's master equations can be derived (Van Hove 1960, Landau and Teller 1936, and Bloch 1957). The procedure of deriving Pauli's equation for the above model is similar to that for a spin-lattice system (Zheng and Schieve 1982).

For the three-level model, Pauli's equation for the probabilities $P(N_1, N_2, N_3; t)$ of N_1, N_2 and N_3 electrons being in level 1, 2 and 3 at time t can be written as

$$\begin{aligned}(\partial/\partial t)P(N_1, N_2, N_3; t) \\ &= q_p(N_3+1)P(N_1-1, N_2, N_3+1; t) + p_p(N_1+1)P(N_1+1, N_2, N_3-1; t) \\ &\quad + q_c(N_3+1)P(N_1, N_2-1, N_3+1; t) + p_c(N_2+1)P(N_1, N_2+1, N_3-1; t) \\ &\quad - (q_pN_3 + q_cN_3 + p_pN_1 + p_cN_2)P(N_1, N_2, N_3; t) \end{aligned} \tag{4}$$

with

$$\begin{aligned} p_p &= (2\pi/\hbar^2)g^2_{\Omega_{31}}\rho_p(\Omega_{31}) \exp(-\beta_p\hbar\Omega_{31}/2)/\sinh(\beta_p\hbar\Omega_{31}/2), \\ q_p &= (2\pi/\hbar^2)g^2_{\Omega_{31}}\rho_p(\Omega_{31}) \exp(\beta_p\hbar\Omega_{31}/2)/\sinh(\beta_p\hbar\Omega_{31}/2), \end{aligned} \tag{5}$$

and similar formulae for p_c and q_c, where $\rho_p(\Omega)$ is the density of states for the pumping photons.

Defining the generating function

$$G(x, y, z; t) = \sum_{N_1,N_2,N_3} P(N_1, N_2, N_3; t)x^{N_1}y^{N_2}z^{N_3}, \tag{6}$$

we have from equation (4)

$$\partial G/\partial t = [q_p(x-z) + q_c(y-z)](\partial G/\partial z) + p_p(z-x)(\partial G/\partial x) + p_c(z-y)(\partial G/\partial y). \tag{7}$$

Its characteristic equations are

$$dx/p_p(z-x) = dy/p_c(z-y) = dz/[q_p(x-z) + q_c(y-z)] = -dt. \tag{8}$$

One first integral can be easily found as

$$\nu_p x + \nu_c y + z = c_0 \tag{9}$$

with

$$\nu_p = q_p/p_p = \exp(\beta_p\hbar\Omega_{31}), \qquad \nu_c = q_c/p_c = \exp(\beta_c\hbar\Omega_{21}). \tag{10}$$

Let

$$U = p_p(\nu_p+1)x + p_p\nu_c y - p_pc_0, \qquad V = p_c\nu_p x + p_c(\nu_c+1)y - p_cc_0. \tag{11a, b}$$

Therefore, from equations (8) and (9) we have

$$dt = dx/U = dy/V = (m\, dx + n\, dy)/(mU + nV). \tag{12}$$

To integrate equations (8) we require m and n to be such that equation (12) can be written in the form (Davis 1960)

$$dt = (m\, dx + n\, dy)/[\lambda(mx + ny) + r]. \tag{13}$$

The equations determining m, n, λ and r from equations (12) and (13) are

$$p_p(\nu_p+1)m + p_c\nu_p n = \lambda m, \tag{14}$$

$$p_p\nu_c m + p_c(\nu_c+1)n = \lambda n, \tag{15}$$

$$r = -(mp_p + np_c)c_0. \tag{16}$$

We obtain

$$\lambda_{1,2}=\tfrac{1}{2}[\![[p_p(\nu_p+1)+p_c(\nu_c+1)]\pm\{[p_p(\nu_p+1)+p_c(\nu_c+1)]^2-4p_pp_c(\nu_p+\nu_c+1)\}^{1/2}]\!], \quad (17)$$

$$m_i=p_c\nu_p, \qquad n_i=\lambda_i-p_p(\nu_p+1) \qquad (i=1,2) \quad (18)\dagger$$

$$r_i=-(m_ip_p+n_ip_c)c_0\equiv k_ic_0\lambda_i. \quad (19)$$

It is easy to verify that both λ_1 and λ_2 are positive. The other two integrals are then given from equation (13) as

$$\exp(-\lambda_it)[\lambda_i(m_ix+n_iy)+r_i]=c_i \qquad (i=1,2). \quad (20)$$

If we assume that at $t=0$ all the N electrons are in level 1 the initial condition for G is then

$$G(x,y,z;0)=x^N. \quad (21)$$

Finally, we obtain

$$G(x,y,z;t)=[\![[(n_1k_2-n_2k_1)(\nu_p+\nu_c+1)]^{-1}\{(n_2/\lambda_1)\exp(-\lambda_1t)[\lambda_1(m_1x+n_1y)+r_1]$$
$$-(n_1/\lambda_2)\exp(-\lambda_2t)[\lambda_2(m_2x+n_2y)+r_2]$$
$$-(n_2k_1-n_1k_2)(\nu_px+\nu_cy+z)\}]\!]^N, \quad (22)$$

where we have used the normalisation condition, i.e. $G(1,1,1;t)=1$. When time t goes to infinity, $G(x,y,z;t)$ approaches

$$G_s(x,y,z)=\left(\frac{\nu_px+\nu_cy+z}{\nu_p+\nu_c+1}\right)^N, \quad (23)$$

which gives in the stationary state

$$\langle N_2\rangle/\langle N_1\rangle=\partial_yG_s/\partial_xG_s|_{x=y=z=1}=\nu_c/\nu_p=\exp[\hbar(\beta_c\Omega_{21}-\beta_p\Omega_{31})]. \quad (24)$$

When the condition $\beta_c\Omega_{21}>\beta_p\Omega_{31}$ holds, then population inversion occurs. To discuss the time-dependent behaviour we can use equation (22). For example, we have

$$\langle N_1(t)\rangle=\frac{N}{\nu_p+\nu_c+1}\left(\nu_p+\frac{1}{n_1k_2-n_2k_1}[n_2m_1\exp(-\lambda_1t)-n_1m_2\exp(-\lambda_2t)]\right), \quad (25)$$

and the variance of N_1

$$\langle N_1^2(t)\rangle-\langle N_1(t)\rangle^2=\partial_x^2G+\partial_xG-(\partial_xG)^2|_{x=y=z=1}=\langle N_1(t)\rangle(1-\langle N_1(t)\rangle/N). \quad (26)$$

Furthermore, correlation functions and higher moments can be calculated in a similar way.

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† m_i and n_i are not uniquely determined, but the arbitrary constant of proportionality can be absorbed into the normalisation factor (see equation (22)).

References

Bloch F 1957 *Phys. Rev.* **105** 1206
Davis H T 1960 *Introduction to Nonlinear Differential and Integral Equations* (Washington: USAEC) p 40
Landau L and Teller E 1936 *Physik Z. Sowjetunion* **10** 34
Rai J and Mehta C L 1982 *Opt. Commun.* **42** 113
Van Hove L 1960 in *Lectures on Statistical Mechanics of Nonequilibrium Phenomena* (New York: Wiley)
Zheng W M and Peng K C 1982 *The double reservoir and negative temperature, preprint*
Zheng W M and Schieve W C 1982 *J. Stat. Phys.* **29** 375